MATH 5061 Lecture on 4/29/2020

Announcement: PS6 due this Friday (May 1).

Take-home final will be available via email & BLACKBOARD on May 6 at 2:30 PM and due via email / BLACKBOARD on May 13 at 2:30 PM.

Spectrum of Riemannian manifolds

(M^m, g) closed, oriented Riem. mfd.

$\rightsquigarrow$ Hodge Laplacian $\Delta := d\delta + \delta d : \Omega^p(M) \to \Omega^p(M)$

$$\mathcal{H}_p := \{ \alpha \in \Omega^p(M) : \Delta\alpha = 0 \} \quad \text{"harmonic } p\text{-forms"}$$

Remark: $\Delta = -\operatorname{div}(\nabla \cdot)$ on $C^\infty(M)$.

More generally, consider the eigenvalue problem:

$$\boxed{\Delta \alpha = \lambda \alpha}$$

λ : eigenvalue

$\alpha \in \Omega^p$: eigenvector

"Functional Analysis" $\Rightarrow \exists$ seq. of eigenvalues

$$\overset{\lambda_0}{\overset{\|}{0}} \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \cdots < \lambda_n < \cdots \longrightarrow +\infty$$

p-Spectrum of (M, g)

and eigenspace $V_n := \{ \alpha \in \Omega^p(M) : \Delta\alpha = \lambda_n \alpha \}$ finite dim'l

$\because \Delta$ self-adj $\Rightarrow \exists$ complete O.N.B. (w.r.t. L^2) $\alpha_1, \alpha_2, \dots$ of eigenforms for $\Omega^p(M)$ for each $p = 0, 1, 2, \dots, m$.

Remark: We know more for the case $p = 0$ (i.e. on functions).

Next: Study the "0"-spectrum of $(S^m, g_{\text{round}}) \subseteq \mathbb{R}^{m+1}$.

(i) When $m = 1$: $S^1 = \bigcirc = \mathbb{R}/_{p \sim p+2\pi}$ $\xRightarrow[\text{analysis}]{\text{Fourier}}$ $\lambda_n = n^2, \ n \geq 0$

$V_n = \operatorname{span}\{\cos n\theta, \sin n\theta\}$

(ii) When $m \geq 2$: $S^m =$  $\subseteq \mathbb{R}^{m+1}$ std coord : $(x^1, x^2, \dots, x^{m+1})$

Spherical coord : $(\underset{|x|}{\overset{\shortparallel}{r}}, \underbrace{\xi^1, \dots, \xi^m}_{\text{angular variable}})$

$$g_{\mathbb{R}^{m+1}} = \sum_{i=1}^{m+1} (dx^i)^2 = dr^2 + r^2 g_{S^m}$$

i.e.
$$(g_{ij}) = \begin{array}{c|c} & \begin{array}{cc} r & \xi^1 \cdots\cdots \xi^m \end{array} \\ \begin{array}{c} r \\ \xi^1 \\ \vdots \\ \xi^m \end{array} & \left(\begin{array}{c|c} 1 & 0 \cdots\cdots 0 \\ \hline 0 & \\ \vdots & r^2 g^{S^m}_{ij}(\xi^1, \dots, \xi^m) \\ 0 & \end{array}\right) \end{array}$$

Using this, one can express Δ in spherical coordinates :

$$f \in C^\infty(\mathbb{R}^{m+1}) \rightsquigarrow \boxed{\Delta_{\mathbb{R}^{m+1}} f = \frac{1}{r^m} \frac{\partial}{\partial r}\left(r^m \frac{\partial f}{\partial r}\right) + \frac{1}{r^2} \Delta_{S^m} f} \quad \cdots\cdots (*)$$

Q : Find all eigenvalues / eigenfunctions of Δ_{S^m} ?

Let $P_n(x^1, \dots, x^{m+1}) = r^n f(\xi)$ be "homogeneous" polynomial.
(of degree n)

(E.g.) $P_2(x, y, z) = xy + yz + xz$)

Suppose P_n is harmonic w.r.t. $\Delta_{\mathbb{R}^{m+1}}$. Then

$$0 = \Delta_{\mathbb{R}^{m+1}} P_n = \frac{1}{r^m} \frac{\partial}{\partial r}\left(r^m \frac{\partial}{\partial r}(r^n f(\xi))\right) + \frac{1}{r^2} \Delta_{S^m}(r^n f(\xi))$$

$$= n(m+n-1) \frac{f(\xi)}{r^{2-n}} + \frac{1}{r^{2-n}} \Delta_{S^m} f(\xi)$$

FACT: All eigenfunctions arise this way!

$$\Rightarrow \quad -\Delta_{S^m} f = n(m+n-1) f$$

i.e. $f(\xi)$ is an eigenfunction of (S^m, g_{S^m}) with $\lambda_n = n(m+n-1)$

$$0 = \lambda_0 < \overset{m}{\overset{\shortparallel}{\lambda_1}} < \overset{2(m+1)}{\overset{\shortparallel}{\lambda_2}} < \cdots < \lambda_n < \cdots\cdots \longrightarrow +\infty$$

$V_0 = \left\{\begin{array}{c} \text{const} \\ \text{fcn} \end{array}\right\}$ $V_1 = \left\{\begin{array}{c} \text{linear} \\ \text{fcn} \end{array}\right\}$ $V_2 = \left\{\begin{array}{c} \text{quadratic} \\ \text{harmonic} \\ \text{polynomials} \end{array}\right\}$ "Spherical harmonics"

Eigenvalue estimates

GOAL: How to estimate (from below / above) the spectrum of (M^m, g) in terms of "geometric information" like curvature, volume, diam.?

Lichnerowicz Thm:

Let (M^m, g) be closed with $Ric \geq (m-1)\kappa g$ for some const. $\kappa > 0$.

Then,
$$\boxed{\lambda_1(M^m, g) \geq m\kappa = \lambda_1\left(S^m\left(\tfrac{1}{\sqrt{\kappa}}\right), g_{round}\right)}$$

Remark: "=" holds $\iff$ $(M^m, g) \overset{\text{isometric}}{\cong} \left(S^m\left(\tfrac{1}{\sqrt{\kappa}}\right), g_{round}\right)$. (Obata)

Basic Idea: Integrate some "Bochner-Weitzenböck formula"

Recall: (WF) for 1-forms $\alpha \in \Omega^1(M)$:

$$\frac{1}{2}\Delta|\alpha|^2 = |D\alpha|^2 + \langle \alpha, \Delta\alpha\rangle + Ric(\alpha^\#, \alpha^\#)$$

Apply this to an "exact" 1-form $\alpha = du$ for $u \in C^\infty(M)$.

$$\frac{1}{2}\Delta|du|^2 = |D(du)|^2 + \langle du, \Delta(du)\rangle + Ric(du^\#, du^\#)$$

i.e.
$$\boxed{\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + Ric(\nabla u, \nabla u)} \quad (\#)$$

$\forall u \in C^\infty(M)$

Proof of Lichnerowicz Thm:

Assume $u \in C^\infty(M)$ is a first eigenfunction of (M^m, g).

i.e. $-\Delta u = \lambda u$ where $\lambda = \lambda_1(M, g)$.

Plug into (#) and integrate over M:

$$0 = \frac{1}{2}\int_M \Delta|\nabla u|^2 = \int_M \overbrace{|\mathrm{Hess}\, u|^2}^{\geq 0} + \langle \nabla u, \nabla(\overbrace{\Delta u}^{-\lambda u})\rangle + \overbrace{\mathrm{Ric}(\nabla u, \nabla u)}^{\geq (m-1)\kappa|\nabla u|^2}$$

div. thm. $\because M$ closed

Note: $\mathrm{Hess}\, u = \mathring{\mathrm{Hess}}\, u + \frac{\Delta u}{m} g$, so we have

$$|\mathrm{Hess}\, u|^2 = |\mathring{\mathrm{Hess}}\, u|^2 + \left|\frac{\Delta u}{m} g\right|^2 \geq \frac{(\Delta u)^2}{m} = \frac{\lambda^2}{m} u^2$$

Putting all these together.

$$0 \geq \int_M \frac{\lambda^2}{m} u^2 - \lambda|\nabla u|^2 + (m-1)\kappa|\nabla u|^2$$

$$\int_M \mathrm{div}(u\nabla u) = \int_M u\Delta u + |\nabla u|^2$$

Observe: $$\lambda\int_M u^2 = \int_M -u\Delta u = \int_M |\nabla u|^2$$

$$\Rightarrow \quad 0 \geq \int_M \underbrace{\left(\frac{\lambda}{m} - \lambda + (m-1)\kappa\right)}_{\text{constant } \leq 0} |\nabla u|^2$$

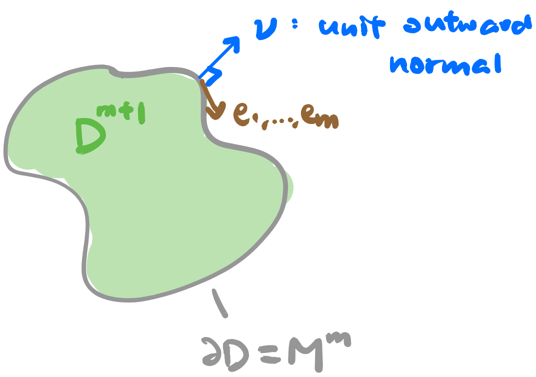
$\because u \not\equiv$ const.

$$\Rightarrow \quad \frac{m-1}{m}\lambda \geq (m-1)\kappa \qquad \text{i.e.} \qquad \lambda \geq m\kappa.$$

$\square$

Reilly's Inequality & applications

Consider: (D^{m+1}, g) cpt Riem. mfd w/. boundary $\partial D = M^m$



$\{e_1, \ldots, e_m\}$ (local) O.N.B for TM

2nd f.f. of $\partial D = M$: $\mathrm{II}(X, Y) := -\langle \bar{\nabla}_X Y, \nu\rangle$

$\xrightarrow{\text{trace}}$ mean curvature H.

In O.N.B., $h_{ij} = \mathrm{II}(e_i, e_j)$. $H = \sum_i h_{ii}$.

Notation: On D^{m+1}, $\bar{\nabla}$, $\bar{\Delta}$

On M^m, ∇, Δ

Reilly's formula: $\forall f \in C^\infty(D)$, we have

$$\frac{m}{m+1}\underbrace{\int_D (\bar\Delta f)^2}_{\text{interior terms}} \geq \int_D \overline{Ric}(\bar\nabla f, \bar\nabla f)$$

$$+ \int_M \underbrace{H\left(\frac{\partial f}{\partial \nu}\right)^2 + 2(\Delta f)\left(\frac{\partial f}{\partial \nu}\right) + \mathrm{II}(\nabla f, \nabla f)}_{\text{boundary terms}}$$

interior terms

Moreover, "=" holds $\iff$ $\overline{Hess}\, f = \dfrac{\bar\Delta f}{m+1}\, g$ in D.

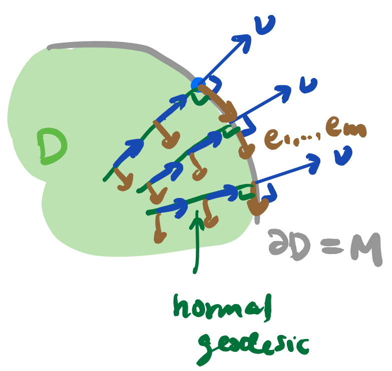
"Sketch of Proof": Integrate (#) over D, $\forall f \in C^\infty(D)$.

$$\underbrace{\int_D \frac{1}{2}\bar\Delta|\bar\nabla f|^2}_{\overset{\text{div. thm.}}{=}\ \frac{1}{2}\int_M \frac{\partial}{\partial\nu}|\bar\nabla f|^2} = \int_D \underbrace{|\overline{Hess}\, f|^2}_{\geq\ \frac{1}{m+1}(\bar\Delta f)^2} + \underbrace{\langle \bar\nabla f, \bar\nabla(\bar\Delta f)\rangle}_{\text{integrate by part}} + \underbrace{\overline{Ric}(\bar\nabla f, \bar\nabla f)}_{\text{keep}}$$

(##)

Compute this!

Establish a local O.N.B.:



Want: Extend O.N.B. $\{e_1, \dots, e_m, e_{m+1} = \nu\}$ locally from M into D s.t.

- $D_{e_{m+1}} e_i \equiv 0 \qquad i = 1, \dots, m+1$
- e_{m+1} tangent to geodesics $\perp$ to M

In these O.N.B., $|\bar\nabla f|^2 = \sum_{i=1}^{m+1} (e_i(f))^2$ in some tubular nbd near M.

So, $\frac{1}{2}\frac{\partial}{\partial\nu}|\bar\nabla f|^2 \overset{e_{m+1} = \nu}{=} \sum_{i=1}^{m+1} e_i(f)\, e_{m+1}(e_i f)$

$$= e_{m+1}(f)\,\underbrace{e_{m+1}(e_{m+1} f)} + \sum_{i=1}^{m} (e_i f)\,\big(e_{m+1}(e_i f)\big)$$

swap

$\cancel{\overline{\nabla}_{e_{m+1}} e_i}^{\,0} - \overline{\nabla}_{e_i} e_{m+1}$

$\|$

$$= e_{m+1}(f)\left(\overline{\Delta} f - \sum_{i=1}^{m} f_{;ii}\right) + \sum_{i=1}^{m} (e_i f)\Big(e_i(e_{m+1} f) + [e_{m+1}, e_i] f\Big)$$

$\|$ $\quad \sum_i f_{;ii} = \sum_i \left(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i} f - \overline{\nabla}_{\overline{\nabla}_{e_i} e_i} f\right)$

$\because \nu = e_{m+1}$

$$= \left(\frac{\partial f}{\partial \nu}\right)\left(\overline{\Delta} f - \Delta f - H\frac{\partial f}{\partial \nu}\right) + \sum_{i=1}^{m} (e_i f)\left(e_i\left(\frac{\partial f}{\partial \nu}\right) - \sum_{j=1}^{m} h_{ij}(e_j f)\right)$$

L.H.S. of (##) $= \displaystyle\int_M (\overline{\Delta} f)\left(\frac{\partial f}{\partial \nu}\right) - (\Delta f)\left(\frac{\partial f}{\partial \nu}\right) - H\left(\frac{\partial f}{\partial \nu}\right)^2 + \left\langle \nabla f, \nabla\left(\frac{\partial f}{\partial \nu}\right)\right\rangle - \mathrm{II}(\nabla f, \nabla f)$

div thm over the closed mfd M

$$= \int_M (\overline{\Delta} f)\left(\frac{\partial f}{\partial \nu}\right) - 2(\Delta f)\left(\frac{\partial f}{\partial \nu}\right) - H\left(\frac{\partial f}{\partial \nu}\right)^2 - \mathrm{II}(\nabla f, \nabla f)$$

R.H.S. of (##) $\geq \displaystyle\int_D \frac{1}{m+1}(\overline{\Delta} f)^2 \overbrace{- \int_D (\overline{\Delta} f)^2 + \int_M (\overline{\Delta} f)\left(\frac{\partial f}{\partial \nu}\right)}^{\text{after integration by part.}} + \int_D \overline{\mathrm{Ric}}(\overline{\nabla} f, \overline{\nabla} f)$

Combining these two,

$$\frac{m}{m+1}\int_D (\overline{\Delta} f)^2 \geq \int_D \overline{\mathrm{Ric}}(\overline{\nabla} f, \overline{\nabla} f) + \int_M H\left(\frac{\partial f}{\partial \nu}\right)^2 + 2(\Delta f)\left(\frac{\partial f}{\partial \nu}\right) + \mathrm{II}(\nabla f, \nabla f)$$

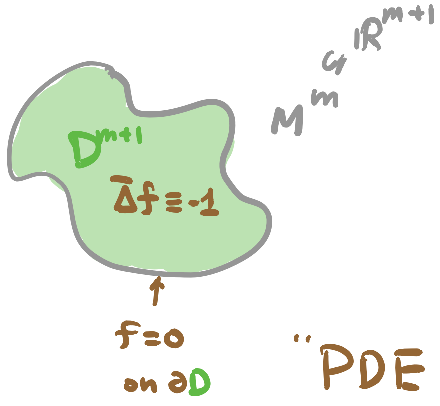
$\blacksquare$

Idea: By choosing some "special functions" in Reilly's formula we can conclude some geometric / topological results.

Alexandrov's Theorem: Any closed embedded hypersurface $M^m \subseteq \mathbb{R}^{m+1}$ with constant mean curvature (CMC) is a round sphere.

Remark: False if M is not closed (Delaunay) or embedded (Wente)

<u>Proof</u>: Jordan–Brouwer separation Thm $\Rightarrow$ M bounds a cpt domain D



<u>WLOG</u>: $H \equiv m$ (by scaling)

<u>Goal</u>: Find a "good" $f \in C^\infty(D)$ and plug into Reilly's formula

"PDE theory" $\Rightarrow$ $\begin{cases} \bar{\Delta} f = -1 & \text{in } D \\ f|_M = 0 \end{cases}$

has a solution $f \in C^\infty(D)$

Plug into Reilly's ineq.,

$$\frac{m}{m+1}\int_D (\underbrace{\bar{\Delta} f}_{=-1})^2 \geq \int_D \overbrace{\overline{Ric}(\bar{\nabla} f, \bar{\nabla} f)}^{=0 \ \because\ D \subseteq \mathbb{R}^{m+1} \text{ flat}}$$

$$+ \int_M \underbrace{H}_{=m}\left(\frac{\partial f}{\partial \nu}\right)^2 + 2(\underbrace{\Delta f}_{=0})\left(\frac{\partial f}{\partial \nu}\right) + \mathrm{II}(\underbrace{\nabla f}_{=0}, \underbrace{\nabla f}_{=0})$$

$$\Rightarrow \quad \frac{m}{m+1}|D| = \frac{m}{m+1}\int_D 1 \geq m \int_M \left(\frac{\partial f}{\partial \nu}\right)^2 \quad \text{——— (1)}$$

$|D|$: Volume of D

By Cauchy-Schwarz ineq.,

$$|D|^2 = \left(\int_D \underbrace{\bar{\Delta} f}_{=-1}\right)^2 \overset{\text{div. thm.}}{=} \left(\int_M \frac{\partial f}{\partial \nu}\right)^2 \overset{\text{C.S.}}{\leq} \left(\int_M 1^2\right)\left(\int_M \left(\frac{\partial f}{\partial \nu}\right)^2\right)$$

$$= |M| \int_M \left(\frac{\partial f}{\partial \nu}\right)^2.$$

$|M|$: area of M

$$\Rightarrow \quad |D|^2 \leq |M| \int_M \left(\frac{\partial f}{\partial \nu}\right)^2 \quad \text{— (2)}$$

(1) & (2) $\Rightarrow$ $\boxed{(m+1)|D| \leq |M|}$

Claim: $(m+1)|D| = |M|$

Proof of Claim: Let $\vec{X}(x) := (x^1, x^2, \dots, x^{m+1})$ position vector field.

- $\Delta_M \vec{X} \underset{(\text{Ex})}{=} \vec{H} \equiv -m\nu$

$$\Rightarrow \frac{1}{2}\Delta_M |\vec{X}|^2 = \langle \Delta_M \vec{X}, \vec{X}\rangle + |\nabla_M \vec{X}|^2$$
$$= \langle \vec{H}, \vec{X}\rangle + m$$

$$\therefore \quad \frac{1}{2}\Delta_M |\vec{X}|^2 = -m\langle \vec{X}, \nu\rangle + m.$$

Integrate over the closed hypersurface M,

$$0 \underset{\substack{\text{div thm} \\ M \text{ closed}}}{=} \frac{1}{2}\int_M \Delta_M |\vec{X}|^2 = -m\int_M \langle \vec{X}, \nu\rangle + m|M|$$

i.e. $$|M| = \int_M \langle \vec{X}, \nu\rangle \underset{\substack{\text{div thm} \\ \text{on } D}}{=} \int_D \operatorname{div}_{\mathbb{R}^{m+1}} \vec{X} = (m+1)|D|$$

$\square_{\text{Claim}}$

Claim $\Rightarrow$ All inequalities in (1), (2) are equalities.

$$\Rightarrow \quad \overline{\text{Hess}}\, f = \frac{-1}{m+1}\delta \quad \Rightarrow \quad \overline{\text{Hess}}\Big(\underbrace{f + \frac{1}{2(m+1)}|\vec{X}|^2}_{\substack{\text{linear function} \\ +\text{ constant}}}\Big) \equiv 0 \text{ in } D \Rightarrow M = \text{round sphere}$$

$\square$

~ END OF COURSE ~